

# Gersten's conjecture

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## Abstract

The purpose of this article is to prove that Gersten's conjecture for a commutative regular local ring is true. As its applications, we will prove the vanishing conjecture for certain Chow groups, generator conjecture for certain  $K$ -groups and Bloch's formula for absolute case.

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## 0 Introduction

The purpose of this note is to prove the following theorem.

**Theorem 0.1** (Gersten's conjecture).

*For any commutative regular local ring  $R$ , Gersten's conjecture is true. That is for any natural numbers  $n, p$ , the canonical inclusion  $\mathcal{M}^{p+1}(R) \hookrightarrow \mathcal{M}^p(R)$  induces the zero map on  $K$ -groups*

$$K_n(\mathcal{M}^{p+1}(R)) \rightarrow K_n(\mathcal{M}^p(R)) ,$$

*where  $\mathcal{M}^i(R)$  is the category of finitely generated  $R$ -modules  $M$  with  $\text{Codim}_{\text{Spec } R} \text{Supp } M \geq i$ .*

Gersten's conjecture is proposed in [Ger73]. More precise historical back grounds of this conjecture are explained in [Moc07]. In §1, we will prove the main theorem and in §2, we will also discuss applications of this conjecture.

**Acknowledgement** The author thankful to Shuji Saito for encouraging him, to Fabrice Orgozozo for stimulating argument about Corollary 2.4, to Takeshi Saito for making him to get to the reduction argument in Lemma 1.3, and to Kazuhiko Kurano for teaching him condition (iii) in Proposition 2.2.

# 1 Proof of the main theorem

From now on, let  $R$  be a commutative regular local ring. Proof of the main theorem is divided series of lemmas. First we will improve Quillen's reduction argument in the proof of Gersten's conjecture in [Qui73].

**Lemma 1.1** (Quillen induction).

*To prove the main theorem, we shall only check the following assertion:*

*For any non-negative integers  $n, p$ , and any regular sequence  $f_1, \dots, f_{p+1}$  in  $R$ , the canonical map induced from the inclusion map  $\mathcal{P}(R/(f_1, \dots, f_{p+1})) \hookrightarrow \mathcal{M}^p(R)$ ,*

$$K_n(\mathcal{P}(R/(f_1, \dots, f_{p+1}))) \rightarrow K_n(\mathcal{M}^p(R))$$

*is zero.*

*Proof.* In the proof of Theorem 5.11 in [Qui73], we have the following formula

$$K_n(\mathcal{M}^{p+1}(R)) = \operatorname{colim}_{\substack{t: \text{regular} \\ \text{element}}} K_n(\mathcal{M}^p(R/tR)).$$

Since  $R$  is UFD by [AB59], for any regular element  $t$  in  $R$ , we can write  $t = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$  where  $p_i$  are prime elements. By dévissage theorem in [Qui73], we have the following formula

$$K_n(\mathcal{M}^p(R/tR)) \xrightarrow{\sim} K_n(\mathcal{M}^p(R/p_1 p_2 \dots p_r R)).$$

**Claim**

We have the following formula

$$K_n(\mathcal{M}^p(R/p_1 p_2 \dots p_r R)) \xrightarrow{\sim} \bigoplus_{i=1}^r K_n(\mathcal{M}^p(R/p_i R)).$$

*Proof of Claim.* We put  $X = \operatorname{Spec} R/p_1 p_2 \dots p_r R$  and  $X_i = \operatorname{Spec} R/p_i R$ . For any closed set  $Y \subset X$ , we put  $\operatorname{Perf}^Y(X)$  the category of strictly perfect complexes which are acyclic on  $X - Y$ . We also put  $\operatorname{Perf}^p(X) := \bigcup_{\substack{Y \subset X \\ \operatorname{Codim} Y \geq p}} \operatorname{Perf}^Y(X)$ . Then we have the following identities

$$\begin{aligned} K_n(\mathcal{M}^p(X)) &\xrightarrow{\text{I}} \operatorname{colim}_{\substack{Y \subset X \\ \operatorname{Codim} Y \geq p}} K'_n(Y) \xrightarrow{\text{II}} \operatorname{colim}_{\substack{Y \subset X \\ \operatorname{Codim} Y \geq p}} K_n(X \text{ on } Y) \\ &\xrightarrow{\sim} \operatorname{colim}_{\substack{Y \subset X \\ \operatorname{Codim} Y \geq p}} \bigoplus_{i=1}^r K_n(X_i \text{ on } X_i \cap Y) \xrightarrow{\text{III}} \bigoplus_{i=1}^r \operatorname{colim}_{\substack{Y \subset X_i \\ \operatorname{Codim} Y \geq p}} K_n(X_i \text{ on } Y) \\ &\xrightarrow{\text{II}} \bigoplus_{i=1}^r \operatorname{colim}_{\substack{Y \subset X_i \\ \operatorname{Codim} Y \geq p}} K'_n(Y) \xrightarrow{\text{I}} \bigoplus_{i=1}^r K_n(\mathcal{M}^p(X_i)) \end{aligned}$$

where the isomorphisms I are proved by continuity [Qui73], [TT90], the isomorphisms II are proved by the Poincaré duality and comparing the following fibration sequences [Qui73] and [TT90]

$$\begin{aligned} K'(Y) &\rightarrow K'(X) \rightarrow K'(X - Y), \\ K(X \text{ on } Y) &\rightarrow K(X) \rightarrow K(X - Y) \end{aligned}$$

for any closed set  $Y \subset X$  and to prove the isomorphism III, we are using the fact that all  $X_i$  are equidimensional.  $\square$

Therefore to prove Gersten's conjecture we shall only check that for any prime element  $f$ , the inclusion map  $\mathcal{M}^p(R/fR) \rightarrow \mathcal{M}^p(R)$  induces the zero map

$$K_n(\mathcal{M}^p(R/fR)) \rightarrow K_n(\mathcal{M}^p(R)).$$

Since  $R/fR$  is regular, inductive argument implies that to prove Gersten's conjecture we shall only check that for any regular sequence  $f_1, \dots, f_{p+1}$  such that  $(f_1, \dots, f_{p+1})$  is prime ideal, the inclusion map  $\mathcal{M}(R/(f_1, \dots, f_{p+1})) \rightarrow \mathcal{M}^p(R)$  induces the zero map

$$K_n(\mathcal{M}(R/(f_1, \dots, f_{p+1}))) \rightarrow K_n(\mathcal{M}^p(R)).$$

Since  $R/(f_1, \dots, f_{p+1})$  is regular, we have  $K_n(\mathcal{P}(R/(f_1, \dots, f_{p+1}))) \xrightarrow{\sim} K_n(\mathcal{M}(R/(f_1, \dots, f_{p+1})))$  by resolution theorem in [Qui73]. Hence we get the result.  $\square$

Now **Lemma 1.1** implies the following assertion by famous Gersten-Sherman argument in [Ger73], [She82] p.240, which is an application of the universal property for algebraic  $K$ -theory associated with semisimple exact categories [She92] Corollary 5.2. From now on let  $\mathcal{F}$  be the category of finite pointed connected CW-complexes and frequently using the notations in [Moc07].

**Lemma 1.2** (Gersten-Sherman reduction argument).

*To prove the main theorem, we shall only check the following assertion:*

*For any  $X \in \mathcal{F}$ , any non-negative integer  $p$ , and any regular sequence  $f_1, \dots, f_{p+1}$  in  $R$ , the canonical map induced from the inclusion map  $\mathcal{P}(R/(f_1, \dots, f_{p+1})) \hookrightarrow \mathcal{M}^p(R)$ ,*

$$\tilde{R}_0(\pi_1(X), \mathcal{P}(R/(f_1, \dots, f_{p+1}))) \rightarrow \tilde{R}_0(\pi_1(X), \mathcal{M}^p(R)) \xrightarrow{\tilde{\text{Sh}}} [X, (\mathbb{K}(\mathcal{M}^p(R)))_0]_*$$

*is zero.*

*Proof.* We have the following commutative diagram for each  $X \in \mathcal{F}$ :

$$\begin{array}{ccc} \tilde{R}_0(\pi_1(X), \mathcal{P}(R/(f_1, \dots, f_{p+1}))) & \longrightarrow & \tilde{R}_0(\pi_1(X), \mathcal{M}^p(R)) \\ \tilde{\text{Sh}} \downarrow & & \downarrow \tilde{\text{Sh}} \\ [X, (\mathbb{K}(\mathcal{P}(R/(f_1, \dots, f_{p+1}))))_0]_* & \longrightarrow & [X, (\mathbb{K}(\mathcal{M}^p(R)))_0]_* \end{array}$$

It is well-known that  $\mathbb{K}(\mathcal{M}^p(R))$  is a  $H$ -space,  $\mathcal{P}(R/(f_1, \dots, f_{p+1}))$  is semi-simple and by the universal property [She92] Corollary 5.2, we learn that we shall only prove the composition

$$\tilde{R}_0(\pi_1(X), \mathcal{P}(R/(f_1, \dots, f_{p+1}))) \rightarrow \tilde{R}_0(\pi_1(X), \mathcal{M}^p(R)) \xrightarrow{\tilde{\text{Sh}}} [X, (\mathbb{K}(\mathcal{M}^p(R)))_0]_*$$

is the zero map for any  $X \in \mathcal{F}$ .  $\square$

Next we will define equivalence relations between morphisms in  $\mathcal{M}^p(R)$  as follows:

For any  $R$ -modules  $M, N$  in  $\mathcal{M}^p(R)$ , and morphisms  $f, g : M \rightarrow N$ , we will declare  $f \sim g$ .

Then  $\mathcal{M}^p(R)$  is an exact category with equivalence relations satisfying the cogluing axiom in the sense of [Moc07]. So we can define the Grothendieck group of lax  $G$ -representations in  $\mathcal{M}^p(R)$ . (For the precise definition, see [Moc07] Definition 3.9).

**Lemma 1.3** (Retraction principle).

*To prove main theorem, we shall only check the following assertion:*

*In the notation **Lemma 1.2**, the canonical map induced from the inclusion map*

$$\mathcal{P}(R/(f_1, \dots, f_{p+1})) \hookrightarrow \mathcal{M}^p(R),$$

$$R_0(G, \mathcal{P}(R/(f_1, \dots, f_{p+1}))) \rightarrow R_0^{lax}(G, \mathcal{M}^p(R))$$

is zero.

*Proof.* We have the following commutative diagram for each  $X \in \mathcal{F}$ :

$$\begin{array}{ccccc} \tilde{R}_0(\pi_1(X), \mathcal{P}(R/(f_1, \dots, f_{p+1}))) & \longrightarrow & \tilde{R}_0(\pi_1(X), \mathcal{M}^p(R)) & \xrightarrow{\tilde{\text{Sh}}} & [X, (\mathbb{K}^e(\mathcal{M}^p(R)))_0]_* \\ & & \downarrow & & \downarrow \text{I} \\ & & R_0^{lax}(\pi_1(X), \mathcal{M}^p(R)) & \xrightarrow{\text{Sh}^{lax}} & [X, (\mathbb{K}^{lax,e}(\mathcal{M}^p(R)))_0]_* \end{array}$$

where the morphism I is a injection by retraction theorem 3.13 in [Moc07]. Hence we get the result.  $\square$

The following argument is one of a variant of weight argument of the Adams operations. (See [Moc07] §1.)

**Lemma 1.4** (Weight changing argument).

The assertion in **Lemma 1.3** is true. Therefore Gersten's conjecture is true.

*Proof.* We put  $B = R/(f_1, \dots, f_p)$ . Let  $G$  be a group and  $(X, \rho_X)$  be a representation in  $\mathcal{P}(B/f_{p+1}B)$ . Since  $B/f_{p+1}B$  is local,  $X$  is isomorphic to  $(B/f_{p+1}B)^{\oplus m}$  for some  $m$  as a  $B/f_{p+1}B$ -module. Then there is a short exact sequence

$$0 \rightarrow B^{\oplus m} \xrightarrow{f_{p+1}} B^{\oplus m} \xrightarrow{\pi} X \rightarrow 0.$$

For each  $g \in G$ , we have a lifting of  $\rho_X(g)$ , that is, a  $R$ -module homomorphism  $\tilde{\rho}(g) : B^{\oplus m} \rightarrow B^{\oplus m}$  such that  $\tilde{\rho}(g) \bmod f_{p+1} = \rho_X(g)$ . Since  $[B^{\oplus m} \xrightarrow{f_{p+1}} B^{\oplus m}]$  is a minimal resolution of  $X$  as a  $B$ -module, (For the definition of a minimal resolution, see [Ser00] p.84.) we can easily learn that  $\tilde{\rho}(g)$  is an isomorphism as a  $B$ -modules by Nakayama's lemma. Therefore  $\tilde{\rho}(g)$  is an isomorphism as a  $R$ -modules. Obviously assignment  $\tilde{\rho} : G \rightarrow \text{Aut}(B^{\oplus m})$  defines a lax representation  $(B^{\oplus m}, \tilde{\rho})$  in  $\mathcal{M}^p(R)$  and we have a short exact sequence

$$(B^{\oplus m}, \tilde{\rho}) \xrightarrow{f_{p+1}} (B^{\oplus m}, \tilde{\rho}) \xrightarrow{\pi} (X, \rho_X)$$

in  $\mathcal{LAX}(\underline{G}, \mathcal{M}^p(R))_s$ . Notice that proving  $f_{p+1}$  is a strict deformation, we need the assumption that  $R$  is commutative!! So we have an identity

$$[(X, \rho_X)] = [(B^{\oplus m}, \tilde{\rho})] - [(B^{\oplus m}, \tilde{\rho})] = 0$$

in  $R_0^{lax}(G, \mathcal{M}^p(R))$ . Hence we get the result.  $\square$

## 2 Corollaries

In this section, we will discuss applications of **Theorem 0.1**. First we get the following absolute version of Bloch's formula.

**Corollary 2.1.**

For a regular noetherian scheme  $X$ , there is a canonical isomorphism

$$H^p(X, \mathcal{K}_p) \xrightarrow{\sim} A^p(X)$$

where  $\mathcal{K}_p$  is the Zariski sheaf on  $X$  associated to the presheaf  $U \mapsto K_n(U)$  and  $A_p(X)$  is defined by the following formula

$$A^p(X) := \text{Coker} \left( \coprod_{x \in X_{p-1}} k(x)^\times \xrightarrow{\text{ord}_x} \coprod_{x \in X_p} \mathbb{Z} \right).$$

Here  $X_i$  is the set of points of codimension  $i$  in  $X$ .

*Proof.* Combining Propositions 5.8 and 5.14 and Remark 5.17 in [Qui73] and **Theorem 0.1**, we can easily obtain the result.  $\square$

Next we will cite the following well-known statement.

**Proposition 2.2.**

(c.f. [Lev85] P.452, Proposition 1.1, [Moc07] Proposition 1.2) *Let  $A$  be a commutative regular local ring. Then the following statements are equivalent.*

- (i) *The maps  $K_0(\mathcal{M}^p(A)) \rightarrow K_0(\mathcal{M}^{p-1}(A))$  are zero for  $p = 1, \dots, \dim A$ .*
- (ii)  *$K_0(\mathcal{M}^p(A))$  is generated by cyclic modules  $A/(f_1, \dots, f_p)$  where  $f_1, \dots, f_p$  forms a regular sequence for  $p = 1, \dots, \dim A$ .*
- (iii)  *$A_p(\text{Spec } A) = 0$  for any  $p < \dim A$ .*

Therefore we get the following results.

**Corollary 2.3** (Vanishing conjecture).

For any commutative regular local ring  $R$  and any  $p < \dim R$ , we have  $A_p(\text{Spec } R) = 0$ .

**Corollary 2.4** (Generator conjecture).

For any commutative regular local ring  $R$ ,  $K_0(\mathcal{M}^p(R))$  is generated by cyclic modules  $R/(f_1, \dots, f_p)$  where  $f_1, \dots, f_p$  forms a regular sequence for  $p = 1, \dots, \dim R$ .

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